On Expressive Description Logics with Composition of Roles in Number Restrictions

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Description Logics

DLs are FO knowledge representation formalisms based on concepts (interpreted as sets of domain objects) and roles (interpreted as binary relations) with a well founded semantics

Different DLs are defined through the constructors they provide to build complex concepts and roles from atomic ones

The "basic" DL $\mathcal{ALC}$ is equipped with conjunction ($C\sqcap$ D), disjunction ($C \sqcup D$) and negation ($\neg C$) of concepts and value restrictions ($\exists R.C, \forall R.C$), which introduce guarded quantifications

For instance, the concept description:

$$\texttt{Men} \sqcap (\neg\exists\texttt{brother} \sqcup \forall\texttt{friend.Female})$$

defines men without brothers or having only female friends

Basic inference problems in DLs

An interpretation is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a domain of individuals and $\mathcal{I}$ is an interpretation function which maps each concept to a subset of $\Delta^{\mathcal{I}}$ and each role to a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$

The concept description C is satisfiable iff there exist an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$ (in this case, we say that $\mathcal{I}$ is a model for C)

The concept description D subsumes the concept description C (written $C \sqsubseteq D$) iff $I^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for all interpretations $\mathcal{I}$

In $\mathcal{ALC}$, subsumption can be reduced to concept satisfiability and *vice versa*: $C \sqsubseteq D$ iff $C \sqcap \neg D$ is unsatisfiable and C is satisfiable iff not $C \sqsubseteq A \sqcap \neg A$, where A is an arbitrary concept name

Introducing number restrictions: $\mathcal{ALCN}$

Number restrictions ($\mathcal{N}$) can be used to restrict the cardinality of the set of fillers of roles (role successors).

For instance, the concept description:

$$\exists^{\leq 3}\texttt{child} \sqcap \forall\texttt{child.Female}$$

defines individuals with at most three daughters and no sons

Expressive extensions $\mathcal{ALCN}(M)$ [Baader & Sattler 1999] allow complex role expressions under number restrictions (e.g. built with constructors $M \subseteq \{\circ, ^+, ^-, \sqcup, \sqcap\}$)

Further extensions

- $\mathcal{ALC}\bar{\mathcal{N}}(M)$ allows the same type of complex role expressions both in value and number restrictions

- $\mathcal{ALCQ}(M)$ allows complex role expressions under qualified number restrictions

For expressiveness and complexity we have the inclusions:

$$\mathcal{ALCN}(M) \subseteq \mathcal{ALC}\bar{\mathcal{N}}(M) \subseteq \mathcal{ALCQ}(M)$$

DLs with role composition ($\circ$) in number restrictions

$\mathcal{ALCN}(\circ)$ allows counting successors of role chains, which can be used to express interesting cardinality constraints on the interrelationships some individuals hold with other objects of the domain

Known decidability results for concept satisfiability [BS99]:

- $\mathcal{ALCN}(\circ)$ is decidable
- $\mathcal{ALCN}(\circ,^+)$ is undecidable
- $\mathcal{ALCN}(\circ,\sqcap)$ is undecidable
- $\mathcal{ALCN}(\circ,^-,\sqcup)$ is undecidable

Two other extensions of $\mathcal{ALCN}(\circ)$

Highly expressive languages (very useful for applications!)

$\mathcal{ALCN}(\circ)$ + inverse roles $\Rightarrow$ $\mathcal{ALC}\bar{\mathcal{N}}(\circ,^{-})$:

$$\texttt{Person} \sqcap \exists\, \texttt{child}^{-} \sqcap \exists^{\leq 1}(\texttt{child}^{-} \circ \texttt{child})$$

(defines persons who are a only child)

$\mathcal{ALCN}(\circ)$ + qualified number restrictions $\Rightarrow$ $\mathcal{ALCQ}(\circ)$:

$$\texttt{Woman} \sqcap \exists^{\geq 3}(\texttt{husband} \circ \texttt{brother}).\texttt{Lawyer}$$

(defines women having at least three lawyers as brother-in-law)

New decidability results

We prove in this work that:

- $\mathcal{ALC}\bar{\mathcal{N}}(\circ,^{-})$ concept satisfiability (and subsumption) is undecidable (via reduction of a domino problem)

- $\mathcal{ALCQ}(\circ)$ concept satisfiability (and subsumption) is decidable (by means of a tableau-based algorithm)

Full proofs can be found in a companion TR available on the Web

The Undecidable Domino Problem

A tiling system $\mathcal{D} = (D, H, V)$ is given by a non-empty set $D = \{D_1, \ldots, D_m\}$ of domino types, and by horizontal and vertical matching pairs $H \subseteq D \times D$, $V \subseteq D \times D$. The domino problem asks for a compatible tiling of the plane $\mathbb{Z} \times \mathbb{Z}$, i.e. a mapping $t : \mathbb{Z} \times \mathbb{Z} \to D$ such that, for all $m, n \in \mathbb{Z}$,

$$\langle\, t(m,n), t(m+1,n) \,\rangle \in H \quad \text{and} \quad \langle\, t(m,n), t(m,n+1) \,\rangle \in V.$$

Reducibility to Concept Satisfiability in DL

A tiling system $\mathcal{D}$ that can be encoded into a DL concept $E_{\mathcal{D}}$ has a compatible tiling iff $E_{\mathcal{D}}$ is satisfiable, i.e. there is an interpretation $\mathcal{I}$ such that $(E_{\mathcal{D}})^{\mathcal{I}} \neq \emptyset$.

The encoding of the domino problem into $E_{\mathcal{D}}$ can be reduced to three subtasks:

- Grid Specification
- Local Compatibility
- Total Reachability

Grid Specification

It must be possible to represent a "square" of $\mathbb{Z} \times \mathbb{Z}$, which consists of points

$$(m, n),\ (m+1, n),\ (m, n+1) \text{ and } (m+1, n+1),$$

in order to yield a complete covering of the plane via a repeating regular grid structure.

The idea is to introduce concepts to represent the grid points and roles to represent the x- and y-successor relationships between points.

Local Compatibility

It must be possible to express that a tiling
is locally compatible, that is
that the x-successor and the y-successor of a point
have an admissible domino type.

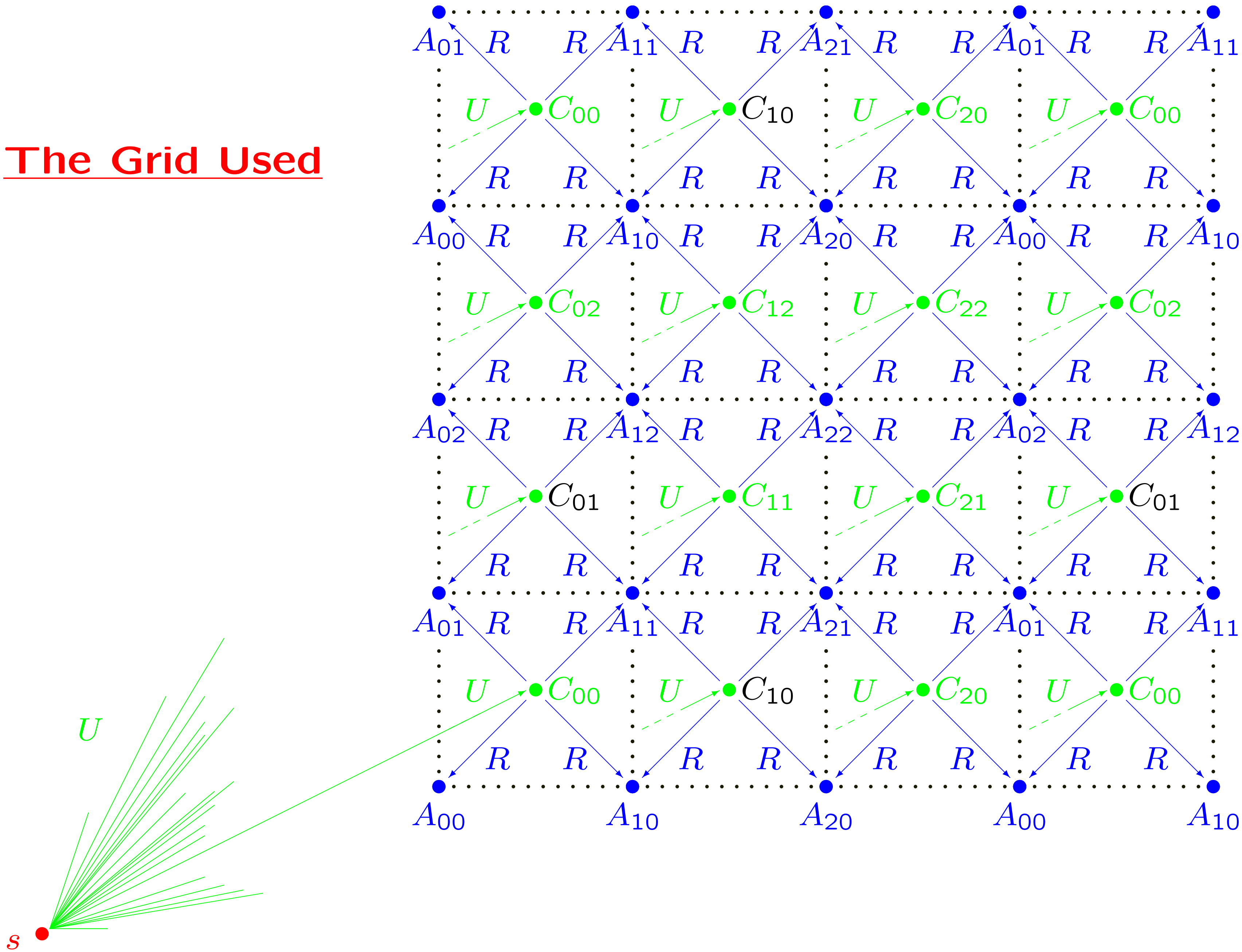
The idea is to associate each domino type D_i
with an atomic concept D_i, and to express
the horizontal and vertical matching conditions
via value restrictions on the stepping roles.

Total Reachability

It must be possible to impose
the above local conditions
on all points in $\mathbb{Z} \times \mathbb{Z}$.

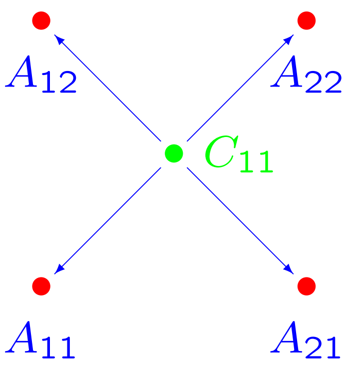
The idea is to construct
a “universal” role and a “start” individual
such that every grid point can be reached
from the start individual.
The local compatibility conditions can then
be globally imposed via value restrictions.

The Grid Used



Grid Enforcement (1)

$$C_{\boxplus} \;:=\; C \sqcap \exists^{\leq 4} R \sqcap \forall R.A_{\boxplus} \sqcap \exists^{\leq 9} R \circ R^{-} \sqcap \bigsqcap_{0 \leq i,j \leq 2} \Big(C_{ij} \Rightarrow (\exists R.A_{ij} \sqcap \exists R.A_{i\oplus 1,j} \sqcap \exists R.A_{i,j\oplus 1} \sqcap \exists R.A_{i\oplus 1,j\oplus 1}) \Big)$$

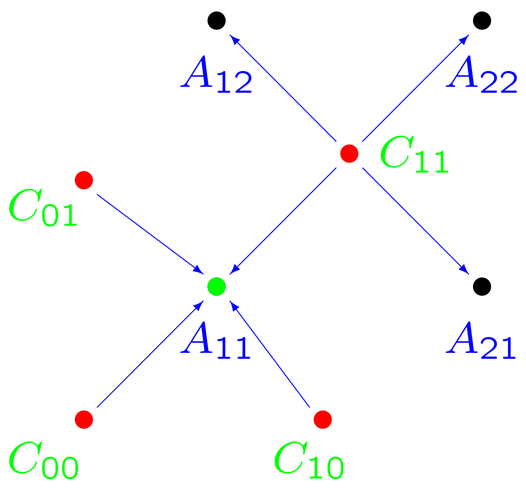


let $c : C_{\boxplus}$, assume $c : C_{11}$

$$C_{11} \;\Rightarrow\; \exists^{\leq 4} R \sqcap \exists R.A_{11} \sqcap \exists R.A_{21} \sqcap \exists R.A_{12} \sqcap \exists R.A_{22}$$

Grid Enforcement (2)

$$A_{\boxplus} \;:=\; A \sqcap \bigsqcap_{0\le i,j\le 2}\Big(A_{ij} \Rightarrow (\exists R^{-}.C_{ij} \sqcap \exists R^{-}.C_{i\oplus 2,j} \sqcap \exists R^{-}.C_{i,j\oplus 2} \sqcap \exists R^{-}.C_{i\oplus 2,j\oplus 2})\Big)$$



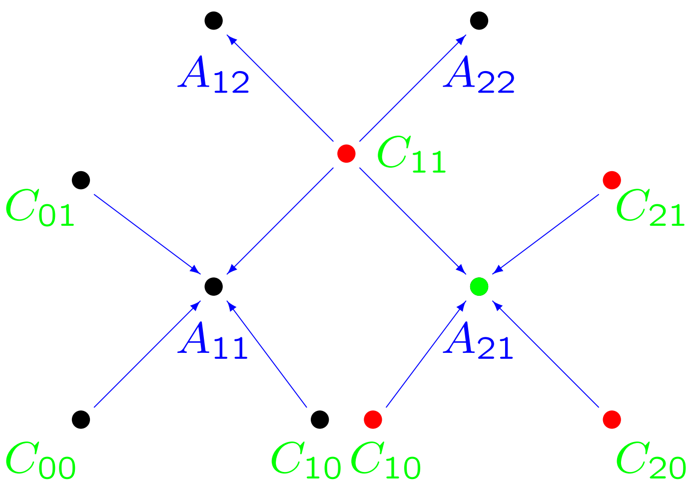
let $a_{11} : A_{11}$, $(c, a_{11}) : R$

$(c : C_{\boxplus} \sqsubseteq \forall R.A_{\boxplus})$

$$A_{11} \;\Rightarrow\; \exists R^{-}.C_{11} \sqcap \exists R^{-}.C_{01} \sqcap \exists R^{-}.C_{10} \sqcap \exists^{-} R.C_{00}$$

Grid Enforcement (2)

$$A_{\boxplus} \;:=\; A \sqcap \bigsqcap_{0 \leq i,j \leq 2} \Big(A_{ij} \Rightarrow (\exists R^{-}.C_{ij} \sqcap \exists R^{-}.C_{i\oplus 2,j} \sqcap \exists R^{-}.C_{i,j\oplus 2} \sqcap \exists R^{-}.C_{i\oplus 2,j\oplus 2}) \Big)$$



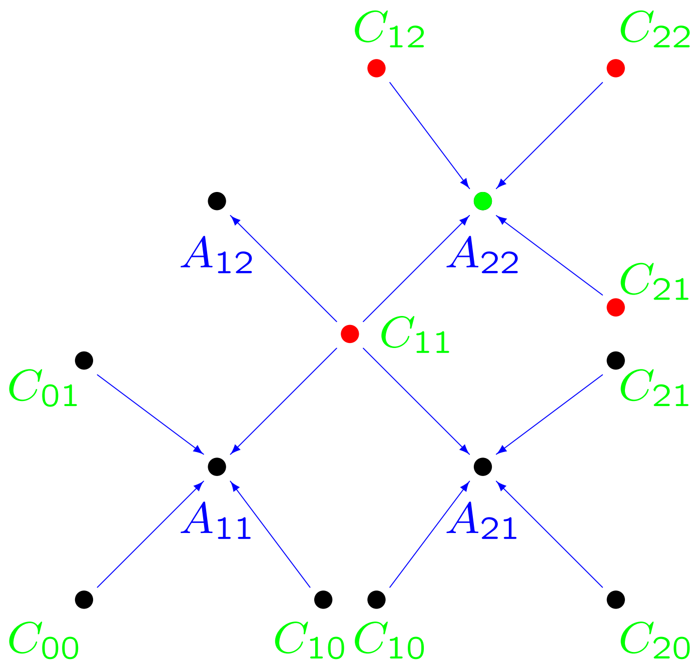
let $a_{21} : A_{21}$, $(c, a_{21}) : R$

$(c : C_{\boxplus} \sqsubseteq \forall R.A_{\boxplus})$

$$A_{21} \;\Rightarrow\; \exists R^{-}.C_{21} \sqcap \exists R^{-}.C_{11} \sqcap \exists R^{-}.C_{20} \sqcap \exists^{-} R.C_{10}$$

Grid Enforcement (2)

$$A_{\boxplus} \;:=\; A \sqcap \bigsqcap_{0\le i,j\le 2}\Big(A_{ij} \Rightarrow (\exists R^{-}.C_{ij} \sqcap \exists R^{-}.C_{i\oplus 2,j} \sqcap \exists R^{-}.C_{i,j\oplus 2} \sqcap \exists R^{-}.C_{i\oplus 2,j\oplus 2})\Big)$$



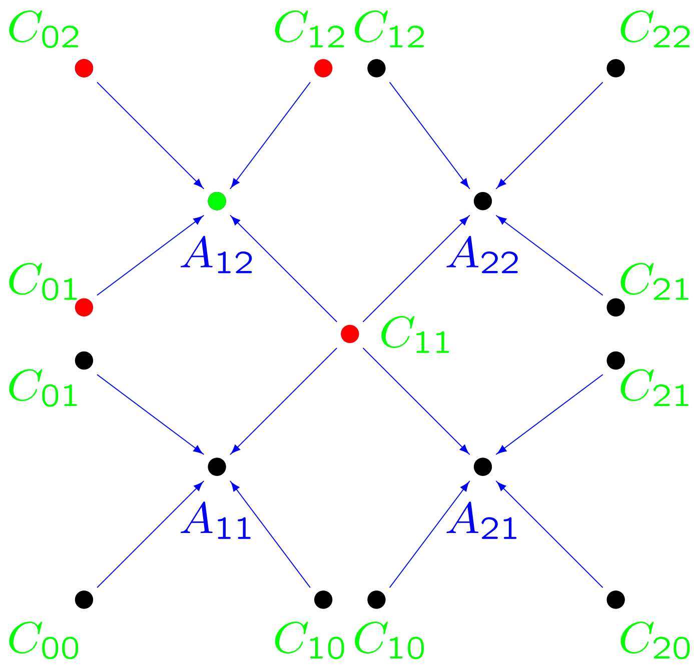
let $a_{22} : A_{22}$, $(c, a_{22}) : R$

$(c : C_{\boxplus} \sqsubseteq \forall R.A_{\boxplus})$

$$A_{22} \;\Rightarrow\; \exists R^{-}.C_{22} \sqcap \exists R^{-}.C_{12} \sqcap \exists R^{-}.C_{21} \sqcap \exists^{-} R.C_{11}$$

Grid Enforcement (2)

$$A_{\boxplus} \ := \ A \sqcap \bigsqcap_{0 \leq i,j \leq 2} \Big(A_{ij} \Rightarrow (\exists R^{-}.C_{ij} \sqcap \exists R^{-}.C_{i\oplus 2,j} \sqcap \exists R^{-}.C_{i,j\oplus 2} \sqcap \exists R^{-}.C_{i\oplus 2,j\oplus 2}) \Big)$$



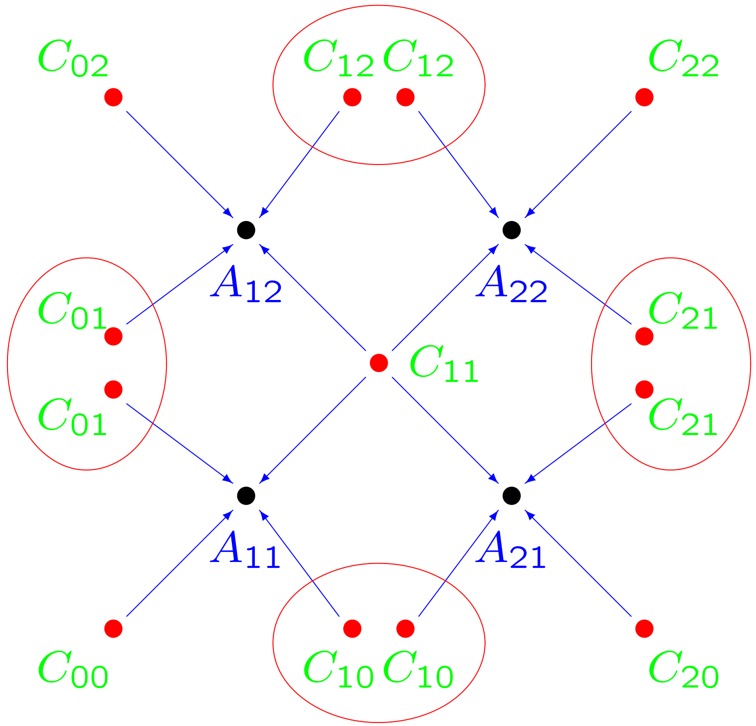
let $a_{12} : A_{12}$, $(c, a_{12}) : R$

$(c : C_{\boxplus} \sqsubseteq \forall R.A_{\boxplus})$

$$A_{12} \ \Rightarrow \ \exists R^{-}.C_{12} \sqcap \exists R^{-}.C_{02} \sqcap \exists R^{-}.C_{11} \sqcap \exists^{-} R.C_{01}$$

Grid Enforcement (3)

$$C_{\boxplus} \;:=\; C \sqcap \exists^{\leq 4} R \sqcap \forall R.A_{\boxplus} \sqcap \exists^{\leq 9} R \circ R^{-} \sqcap$$
$$\bigsqcap_{0 \leq i,j \leq 2} \Big(C_{ij} \Rightarrow (\exists R.A_{ij} \sqcap \exists R.A_{i \oplus 1,j} \sqcap \exists R.A_{i,j \oplus 1} \sqcap \exists R.A_{i \oplus 1, j \oplus 1}) \Big)$$



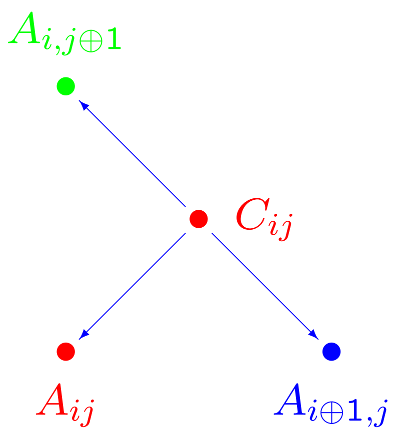
- all C_{ij}s are disjoint
- c has $(R \circ R^{-})$-successors in all nine C_{ij}s
- then c has exactly nine $(R \circ R^{-})$-successors,
- one for each C_{ij}

Local Compatibility

$$C_{\mathcal{D}} :=$$

$$\forall R.\Big(\bigsqcup_{1\leq k\leq m}(D_k \sqcap (\bigsqcap_{\substack{1\leq \ell\leq m\\ k\neq\ell}} \neg D_\ell))\Big) \sqcap \bigsqcap_{0\leq i,j\leq 2}\Big(C_{ij} \Rightarrow \bigsqcap_{1\leq k\leq m} \exists R.(A_{ij} \sqcap D_k)$$

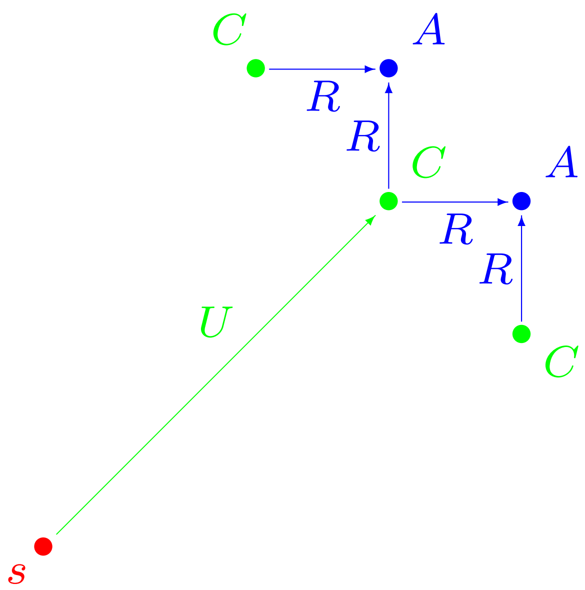
$$\Rightarrow \big(\exists R.(A_{i\oplus 1,j} \sqcap (\bigsqcup_{(D_k,D_\ell)\in H} D_\ell)) \sqcap \exists R.(A_{i,j\oplus 1} \sqcap (\bigsqcup_{(D_k,D_\ell)\in V} D_\ell))\big)\Big)$$



is easily enforced from grid centers: for an (i,j)-type grid cell, the bottom left vertex belongs to A_{ij} and its x- and y-successors to $A_{i\oplus 1,j}$ and $A_{i,j\oplus 1}$, respectively

Total Reachability

$$E_{\mathcal{D}} \;:=\; \exists U \circ R \sqcap \exists^{\leq 1}(U \circ R) \circ (U \circ R)^{-} \sqcap \forall U.\forall R.\forall R^{-}.\exists U^{-} \sqcap \forall U.(C_{\boxplus} \sqcap C_{\mathcal{D}})$$

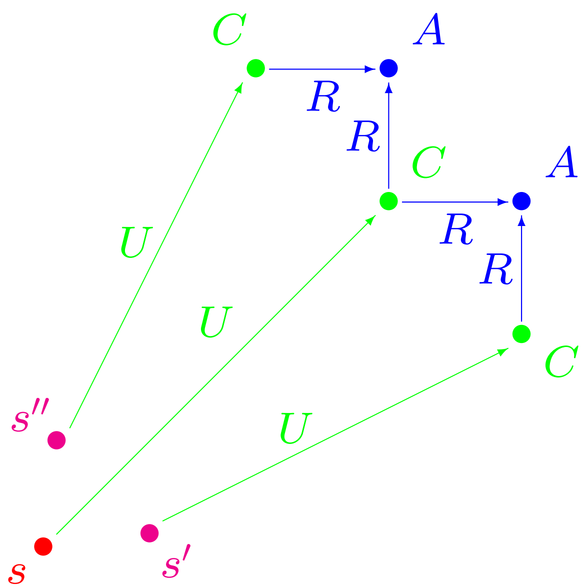


let s : $E_{\mathcal{D}}$ be the starting point

s is connected through U to a grid center, which is connected to the four neighboring grid points through R and to the eight neighboring grid centers through $R \circ R^{-}$

Total Reachability

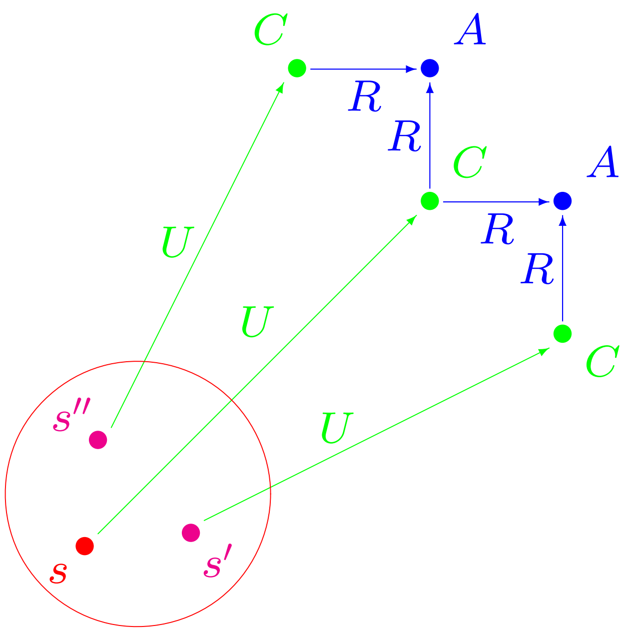
$$E_{\mathcal{D}} \;:=\; \exists U \circ R \sqcap \exists^{\leq 1}(U \circ R) \circ (U \circ R)^{-} \sqcap \forall U.\forall R.\forall R^{-}.\exists U^{-} \sqcap \forall U.(C_{\boxplus} \sqcap C_{\mathcal{D}})$$



all these neighboring grid centers must have U predecessors (e.g. s' and s'')

Total Reachability

$$E_{\mathcal{D}} \;:=\; \exists U \circ R \sqcap \exists^{\leq 1}(U \circ R) \circ (U \circ R)^{-} \sqcap \forall U.\forall R.\forall R^{-}.\exists U^{-} \sqcap \forall U.(C_{\boxplus} \sqcap C_{\mathcal{D}})$$



these predecessors must coincide with s

therefore, also the neighboring grid centers can be reached from the starting point through U

$\mathcal{ALC}\bar{\mathcal{N}}(\circ,^{-})$ is undecidable

The undecidable domino problem has been encoded into an $\mathcal{ALC}\bar{\mathcal{N}}(\circ,^{-})$ concept: $E_{\mathcal{D}}$ is satisfiable iff the tiling system $\mathcal{D}$ has a compatible tiling

Hence, satisfiability of concepts is undecidable for $\mathcal{ALC}\bar{\mathcal{N}}(\circ,^{-})$

Tableau-based decision algorithms for DLs

Based on manipulation of constraints in the form of ABoxes

An ABox $\mathcal{A}$ is a finite set of assertions of the form:

$$\begin{aligned} C(a) &\quad \text{— concept assertion} \\ R(a,b) &\quad \text{— role assertion} \\ a \neq b &\quad \text{— inequality assertion} \end{aligned}$$

where a, b are individual names

The ABox $\mathcal{A}$ is consistent iff it has a model

The individual a is an instance of the concept description C iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$ for all models $\mathcal{I}$ of $\mathcal{A}$.

Tableau-based decision algorithms for DLs (2)

Starting from the initial ABox $\{C(x_0)\}$ the algorithm applies completion rules, which modify the ABox, and stops when no rule can be applied (an ABox is complete iff none of the rules is any longer applicable).

The algorithm says "C is satisfiable" iff a complete and clash-free ABox has been generated.

An ABox $\mathcal{A}$ contains a clash (for $\mathcal{ALCQ}(\circ)$) iff:

- $\{A(x), \neg A(x)\} \subseteq \mathcal{A}$, or
- $(\exists^{\leq n} R_1 \circ \cdots \circ R_m.C)(x) \in \mathcal{A}$ and x has more than n distinct $(R_1 \circ \cdots \circ R_m)$-successors in $\mathcal{A}$ which are instances of C

A Tableau for $\mathcal{ALCQ}(\circ)$ concept satisfiability

- The $\sqcap$-rule and the $\sqcup$-rule are the same as for $\mathcal{ALCN}(\circ)$
- The $\geq$ -rule and $\leq$ -rule of $\mathcal{ALCN}(\circ)$ are modified to take into account the "qualifying" conditions
- A ©-rule (*choose* rule) is added to correctly cope with qualified number restrictions

The proposed extensions making the $\mathcal{ALCN}(\circ)$ algorithm an $\mathcal{ALCQ}(\circ)$ algorithm are very similar to the ones which turn the $\mathcal{ALCN}$ algorithm into an $\mathcal{ALCQ}$ algorithm

Decidability of $\mathcal{ALCQ}(\circ)$ - The Tableau (1)

$\sqcap$-rule, $\sqcup$-rule: the same as for $\mathcal{ALCN}(\circ)$ (and $\mathcal{ALC}$)

$\geq$ -rule:

if 1. $(\exists^{\geq n} R_1 \circ \cdots \circ R_m.C)(x) \in \mathcal{A}$ and

2. x has exactly p $(R_1 \circ \cdots \circ R_m)$-successors $y_1, \ldots, y_p$ with $p < n$ such that $\{C(y_i) \mid 1 \leq i \leq p\} \cup \{y_i \neq y_j \mid 1 \leq i < j \leq p\} \subseteq \mathcal{A}$

then $\mathcal{A}' := \mathcal{A} \cup \{R_1(x, z_{i2}), R_2(z_{i2}, z_{i3}), \ldots, R_m(z_{im}, z_i), C(z_i) \mid 1 \leq i \leq n - p\}$
$\cup \{z_i \neq z_j \mid 1 \leq i < j \leq n - p\} \cup \{y_i \neq z_j \mid 1 \leq i \leq p, 1 \leq j \leq n - p\}$

Decidability of $\mathcal{ALCQ}(\circ)$ - The Tableau (2)

$\leq$ -rule:

if 1. $(\exists^{\leq n} R_1 \circ \cdots \circ R_m.C)(x) \in \mathcal{A}$ and
2. x has more than n $(R_1 \circ \cdots \circ R_m)$-successors $y_1, \ldots, y_p$ such that $\{C(y_i) \mid 1 \leq i \leq p\} \subseteq \mathcal{A}$ and $\{y_i \neq y_j\} \cap \mathcal{A} = \emptyset$

then for some pair y_i, y_j $(1 \leq i < j \leq p)$ such that $\{y_i \neq y_j\} \cap \mathcal{A} = \emptyset$
$\mathcal{A}' := [y_i/y_j]\mathcal{A}$ (i.e. replace in $\mathcal{A}'$ each occurrence of y_i by y_j)

©-rule

if 1. $(\exists^{\bowtie n} R_1 \circ \cdots \circ R_m.C)(x) \in \mathcal{A}$ and
2. y is an $(R_1 \circ \cdots \circ R_m)$-successor of x with $\{C(y), \sim C(y)\} \cap \mathcal{A} = \emptyset$

then $\mathcal{A}' := \mathcal{A} \cup \{D(y)\}$ for some $D \in \{C, \sim C\}$

$\mathcal{ALCQ}(\circ)$ is decidable

Let C_0 be an $\mathcal{ALCQ}(\circ)$-concept (in NNF$^{\bowtie}$) and let $\mathcal{A}$ be an ABox obtained by applying the completion rules to $\{C_0(x_0)\}$. Then

1. The application of the completion rules to $\mathcal{A}$ preserves its models for each interpretation $\mathcal{I}$

2. If $\mathcal{A}$ is complete, $\mathcal{A}$ is clash-free iff it has a model

3. The completion algorithm terminates

4. The size of $\mathcal{A}$ is exponential in the size of C_0

Hence, satisfiability of $\mathcal{ALCQ}(\circ)$ is decidable (in NExpTime)

Conclusions

Known results

$\mathcal{ALC}\bar{\mathcal{N}}(\circ)$ is decidable
$\mathcal{ALC}\bar{\mathcal{N}}(\circ, \sqcap)$, $\mathcal{ALC}\bar{\mathcal{N}}(\circ, ^-, \sqcup)$ and $\mathcal{ALC}\bar{\mathcal{N}}(\circ, ^+)$ are undecidable

We have shown that

$\mathcal{ALC}\bar{\mathcal{N}}(\circ, ^-)$ (and $\mathcal{ALCQ}(\circ, ^-)$) is undecidable
$\mathcal{ALCQ}(\circ)$ (and $\mathcal{ALC}\bar{\mathcal{N}}(\circ)$) is decidable

Still open problems

(un)decidability of $\mathcal{ALCN}(\circ, \sqcup)$, "pure" $\mathcal{ALCN}(\circ, ^-)$
exact complexity of $\mathcal{ALCN}(\circ)$, $\mathcal{ALCQ}(\circ)$ and $\mathcal{ALC}\bar{\mathcal{N}}(\circ)$